Hw1 solution

3. In class, we introduced a simple model of gene expression:

$g \rightarrow m \rightarrow p$

Here g is the gene (DNA), m is messenger RNA, and p is the protein and reactions occur at rates α_1 and α_2 for the first and second reaction, respectively. (The complete model also contains terms for degradation of protein and mRNA.)

(a) Now, assume that there is a small RNA (S) which can hybridize to the messenger RNA (m), and degrade it.

(1) Write a reaction equation for this system assuming that the small RNA s (i) is itself degraded or (ii) is not degraded in the process.

(i)

$$g \xrightarrow{\alpha_1} g + m$$
$$m \xrightarrow{r_m} \varnothing$$
$$m \xrightarrow{\alpha_2} m + p$$
$$p \xrightarrow{r_p} \varnothing$$
$$m + s \xrightarrow{k_1} C$$
$$C \xrightarrow{k_2} \varnothing$$

(ii)

$$g \xrightarrow{\alpha_1} g + m$$
$$m \xrightarrow{r_m} \varnothing$$
$$m \xrightarrow{\alpha_2} m + p$$
$$p \xrightarrow{r_p} \varnothing$$
$$m + s \xrightarrow{k_1} C$$
$$C \xrightarrow{k_2} s$$

(2) Derive the differential equations for both systems

(i)

$$[\dot{g}] = 0$$
$$[\dot{m}] = \alpha_1[g] - r_m[m] - k_1[m][s]$$
$$[\dot{p}] = \alpha_2[m] - r_p[p]$$
$$[\dot{s}] = -k_1[m][s]$$
$$[\dot{C}] = k_1[m][s] - k_2[C]$$

(ii)

$$\dot{[g]} = 0$$
$$\dot{[m]} = \alpha_1[g] - r_m[m] - k_1[m][s]$$
$$\dot{[p]} = \alpha_2[m] - r_p[p]$$
$$\dot{[s]} = -k_1[m][s] + k_2[C]$$
$$\dot{[C]} = k_1[m][s] - k_2[C]$$

(b) A transcriptional repressor is a protein that can inhibit transcription by binding to a gene. Assume that the protein p in the model above is a repressor that can bind to gene g.

(1) What chemical reactions can you use to model such a feedback system?

(2) Write down the differential equations for this system

(1)

$$g \xrightarrow{\alpha_1} g + m$$
$$m \xrightarrow{r_m} \varnothing$$
$$m \xrightarrow{\alpha_2} m + p$$
$$p \xrightarrow{r_p} \varnothing$$
$$p + g \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C$$

(2)

$$\dot{[g]} = -k_1[p][g] + k_{-1}[C]$$
$$\dot{[m]} = \alpha_1[g] - r_m[m]$$
$$\dot{[p]} = \alpha_2[m] - r_p[p] - k_1[p][g] + k_{-1}[C]$$

4. Solving differential equations.

(a) A catalytic reaction is modeled as follows:

A+B→A+C,

$A(t=0)=A_0$, $B(t=0)= B_0$

Write the differential equations for this system, solve them, and plot B as a function of time

$$\frac{d[A]}{dt} = -k[A][B] + k[A][B] = 0$$
$$\frac{d[B]}{dt} = -k[A][B]$$
$$\frac{d[C]}{dt} = k[A][B]$$

Since d[A]/dt = 0, then $[A] = A_0$

$$\frac{d[B]}{dt} = -kA_0[B]$$
$$--> \frac{d[B]}{[B]} = -kA_0 dt$$
$$--> \int \frac{d[B]}{[B]} = -\int kA_0 dt$$
$$--> \ln[B] = -kA_0 t + C$$
$$--> [B] = e^{-kA_0 t + C}$$
$$t = 0, [B] = B_0$$
$$--> B_0 = e^C$$
$$--> [B] = B_0 e^{-kA_0 t}$$

(b) An auto-catalyst system is modeled as follows:

A+B→2A

A(t=0)=A_0, B(t=0)=B_0

Write the differential equations for this system, solve them, and plot B as a function of time.

$$\frac{d[A]}{dt} = -k[A][B] + 2k[A][B] = k[A][B]$$
$$\frac{d[B]}{dt} = -k[A][B]$$

By conservation law, $[A] + [B] = A_0 + B_0$

➔ $[A] = A_0 + B_0 - [B]$

$$\frac{d[B]}{dt} = -k(A_0 + B_0 - [B])[B]$$

➔ $$\frac{d[B]}{(A_0 + B_0 - [B])[B]} = -kdt$$

Let's define $S_0 = A_0 + B_0$

Then the above equation becomes

$$\frac{d[B]}{[B](S_0 - [B])} = -kdt$$

$\rightarrow \frac{d[B]}{[B]}+\frac{d[B]}{(S_0-[B])}=-kS_0dt$

$\rightarrow \int\frac{d[B]}{[B]}+\int\frac{d[B]}{(S_0-[B])}=-\int kS_0dt$

$\rightarrow \ln[B]-\ln(S_0-[B])+=-kS_0t+C$

$\rightarrow [B]/(S_0-[B])=C'e^{-kS_0t}$

Apply the initial condition to the above equation

t=0$\rightarrow [B]=B_0$

$$\rightarrow \qquad C'=\frac{B_0}{S_0-B_0}=\frac{B_0}{A_0}$$

And then $[B]/(S_0-[B])=\frac{A_0}{B_0}e^{-kS_0t}$

Substitute $S_0=A_0+B_0$ into the above equation

$$[B]/(A_0+B_0-[B])=\frac{A_0}{B_0}e^{-k(A_0+B_0)t}$$

By isolating [B] into one side, you should be able to get [B]

$$[B]=\frac{A_0+B_0}{1+\frac{A_0}{B_0}e^{k(A_0+B_0)t}}$$

$$[A]=A_0+B_0-[B]$$

$$\rightarrow [A]=\frac{A_0+B_0}{1+\frac{B_0}{A_0}e^{-k(A_0+B_0)t}}$$